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MULTISTAGE ESTIMATION OF DYNAMICAL AND COLORED NOISE STATES IN --ETC(U)
1977 H D WASHBURN, J M MENDEL AFOSR-75-2797

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REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER: AFOSR-TR- 78-0144	2. GOVT ACCESSION NO.
3. RECIPIENT'S CATALOG NUMBER	
4. TITLE (and Subtitle): MULTISTAGE ESTIMATION OF DYNAMICAL AND COLORED NOISE STATES IN CONTINUOUS-TIME LINEAR SYSTEMS	5. TYPE OF REPORT & PERIOD COVERED: Interim rept.,
	6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s): H. D. Washburn ~~and~~ J. M. Mendel	8. CONTRACT OR GRANT NUMBER(s): AFOSR-75-2797
9. PERFORMING ORGANIZATION NAME AND ADDRESS: University of Southern California, Department of Electrical Engineering-Systems, Los Angeles, CA 90007	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: 61102F 2304 A1
11. CONTROLLING OFFICE NAME AND ADDRESS: Air Force Office of Scientific Research/NM, Bolling AFB, DC 20332	12. REPORT DATE: 1977
	13. NUMBER OF PAGES: 23 (24p.)
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)	15. SECURITY CLASS. (of this report): UNCLASSIFIED
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

* is devised (use twice)

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

In this paper we demonstrate that it is possible to extend Friedland's bias estimation technique, as recently rederived in a constructive manner by Mendel and Washburn to the problem of estimating dynamical states and colored noise states. We have shown how to obtain an exact multistage decomposition not only for the state estimation equations, but also for the associated error covariance equations. Additionally, ~~we have obtained~~ a second-order suboptimal multistage estimator * using a perturbation technique. Whereas a high-order matrix Riccati equation must be solved when the exact results are used, a matrix Riccati equation, of the dimension of the colored noise states, must be solved when the

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20. Abstract

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(THEORY PAPER IN ESTIMATION)

AFOSR-TR- 78-0144

Multistage Estimation of Dynamical and Colored Noise States in Continuous-Time Linear Systems*

by

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ABSTRACT

In this paper we demonstrate that it is possible to extend Friedland's (Ref. 1) bias estimation technique, as recently rederived in a constructive manner by Mendel and Washburn (Ref. 8), to the problem of estimating dynamical states and colored noise states. We have shown how to obtain an exact multistage decomposition not only for the state estimation equations, but also for the associated error covariance equations. Additionally, we have obtained a second-order sub-optimal multistage estimator, using a perturbation technique. Whereas a high-order matrix Riccati equation must be solved when the exact results are used, a matrix Riccati equation, of the dimension of the colored noise states, must be solved when the sub-optimal results are used.

*This work was performed at the University of Southern California, Los Angeles, California, and supported by the Air Force Office of Scientific Research under Grant AFOSR-75-2797. The first author wishes to thank the Hughes Fellowship Program for his support.

I. INTRODUCTION

In this paper, we extend Friedland's (Ref. 1) bias estimation technique, as recently rederived in a constructive manner by Mendel and Washburn (Ref. 8), to the important problem of estimating dynamical* states, $\underline{x}(t)$, and colored noise states, $\underline{z}(t)$, for the following system:

$$S\begin{cases} \dot{\underline{x}} = A\underline{x} + B\underline{z} + \underline{u} \quad ; \quad \underline{x}(0) & (1) \\ \dot{\underline{z}} = c\underline{z} + \varepsilon\underline{w} \quad ; \quad \underline{z}(0) \; ; \; 0 \leq \varepsilon \leq 1 & (2) \\ \underline{y}(t) = H\underline{x} + \underline{v} & (3) \end{cases}$$

We do not show the explicit dependence of vector and/or matrix quantities on time, for notational simplicity. In S, $\underline{x}\epsilon R^n$, $A\epsilon R^{n\times n}$, $C\epsilon R^{r\times r}$, $\underline{u}\epsilon R^n$, $\underline{z}\epsilon R^r$, $\underline{w}\epsilon R^r$, $B\epsilon R^{n\times r}$, $y\epsilon R^s$, $H\epsilon R^{s\times n}$, $\underline{v}\epsilon R^s$; $\underline{x}(0)$ and $\underline{z}(0)$ are independent gaussian random variables; and, $\underline{u}$, $\underline{w}$, and $\underline{v}$ are gaussian white noise processes, for which

$$E\{\underline{u}\} = \underline{0},\ E\{\underline{w}\} = \underline{0},\ \text{and } E\{\underline{v}\} = 0 \tag{4}$$

$$E\{\underline{u}(t)\underline{u}'(\tau)\} = Q_1\delta(t-\tau) \tag{5}$$

$$E\{\varepsilon\underline{w}(t)\underline{w}'(\tau)\varepsilon\} = \varepsilon^2 Q_2\delta(t-\tau) \tag{6}$$

$$E\{\underline{u}(t)\underline{w}'(\tau)\} = 0 \tag{7}$$

$$E\{\underline{v}(t)\underline{v}'(\tau)\} = R\delta(t-\tau) \tag{8}$$

and**

$$\underline{u} \perp \underline{v},\ \underline{w} \perp \underline{v} \tag{9}$$

Colored noise states, $\underline{z}(t)$, are described by a first-order Markov process which is not affected by the dynamical $\underline{x}$-states. Scalar parameter ε, which ranges between

---

* Strictly speaking, $\underline{x}(t)$ and $\underline{z}(t)$ are both dynamical state vectors; however, since it has become customary to refer to $\underline{z}(t)$ as colored noise, we shall distinguish between $\underline{x}(t)$ and $\underline{z}(t)$ as indicated.

**$\underline{a} \perp \underline{b} \Rightarrow E\{\underline{a}\underline{b}'\} = 0$.

zero and unity, is a useful artifice which permits us to reduce our results to earlier results for which $\varepsilon = 0$, in which cases, $\underline{z}(t)$ can be thought of as a bias (a constant bias if $C = 0$). It also permits us to make the transition from truly constant biases to "almost" constant biases. Finally, we wish to emphasize the fact that colored noise disturbances are quite common in practical applications. In a launch vehicle application, for example, $\underline{z}(t)$ would be a finite-bandwidth wind process.

In the sequel, we will show that the optimum filtered estimate of $\underline{x}$, $\hat{\underline{x}}$, and its associated error covariance matrix P_x, can be expressed as

$$\hat{\underline{x}} = \tilde{\underline{x}} + V\hat{\underline{z}} + \hat{\underline{\xi}} \tag{10}$$

$$P_x = P_{x_1} + P_1 + VP_{12}' + P_{12}V' + VP_zV' \tag{11}$$

where $\tilde{\underline{x}}$ is the colored-noise-free estimate of $\underline{x}$, computed as if no $\underline{z}$ states were present in S; $\hat{\underline{z}}$ is the optimum estimate of the colored noise states; $\hat{\underline{\xi}}$ is the estimate of an $n \times 1$ residual random process; and V is a matrix which blends the estimates $\tilde{\underline{x}}$ and $\hat{\underline{z}}$ together with $\hat{\underline{\xi}}$ to give the <u>colored-noise corrected estimate</u> of $\underline{x}$, $\hat{\underline{x}}$. Matrices P_{x_1}, P_1, P_{12}, and P_z are defined in Section II.

The multistage decomposition in Eqs. (10) and (11) is an extension of the results presented by Friedland (Ref. 1), which were recently rederived by Mendel and Washburn (Ref. 8) in a constructive manner — constructive in the sense that their derivation can be applied to more difficult situations, such as the one considered in the present paper. Friedland considered the case where $\varepsilon = 0$ and $C = 0$, so that $\underline{z}$ is a bias vector. Tacker and Lee (Ref. 2) extended Friedland's results to the case where just $\varepsilon = 0$ (i.e., time-varying biases). Tanaka (Ref. 3) treats the case we are considering in this paper, but for discrete-time systems; however, his results are stated without proof, and, it is not at all clear how he obtained them. A full-blown generalization of Friedland's results to partitioned

dynamical system, where $\underline{x}$ and $\underline{z}$ state equations are completely coupled, is given in Washburn (Ref. 4) and will be reported on in a later publication.

Our multistage filtering results, for colored noise and dynamical states, are given in Section II. These results are also compared with the less general results of Tacker and Lee, and Friedland. Suboptimal second-order filtering algorithms, obtained via a perturbation technique, are described in Section III. Proofs for all theorems are given in the appendices.

II. EXACT MULTISTAGE FILTERING RESULTS

Our main results are stated in the following:

Theorem 1. For system S, if $P_{xz}(0) = 0$, then the multistage minimum-variance filter estimator-equations are:

$$\hat{\underline{x}} = \tilde{\underline{x}} + V\hat{\underline{z}} + \hat{\underline{\xi}} \tag{12}$$

$$\dot{\hat{\underline{z}}} = C\hat{\underline{z}} + K_z[\underline{y}(t) - H\hat{\underline{\xi}} - HV\hat{\underline{z}} - H\tilde{\underline{x}}] \;;\; \hat{\underline{z}}(0) \tag{13}$$

$$\dot{\hat{\underline{\xi}}} = (A - K_{x_1}H)\hat{\underline{\xi}} + K_\xi[\underline{y}(t) - H\hat{\underline{\xi}} - HV\hat{\underline{z}} - H\tilde{\underline{x}}] \;;\; \hat{\underline{\xi}}(0) \tag{14}$$

and

$$\dot{\tilde{\underline{x}}} = (A - K_{x_1}H)\tilde{\underline{x}} + K_{x_1}[\underline{y}(t) - H\tilde{\underline{x}}] \;;\; \tilde{\underline{x}}(0) \tag{15}$$

where the gains are

$$\dot{V} = (A - K_{x_1}H)V + B - VC \;;\; V(0) = 0 \tag{16}$$

$$K_z = (P_{12}' + P_zV')H'R^{-1} \tag{17}$$

$$K_\xi = (P_1 + P_{12}V')H'R^{-1} \tag{18}$$

and

$$K_{x_1} = P_{x_1}H'R^{-1} \tag{19}$$

Error covariance matrix $\bar{P}$, where

$$\bar{P} = \begin{bmatrix} P_1 & P_{12} \\ P_{12}' & P_2 \end{bmatrix}, \tag{20}$$

is computed from

$$
\left.
\begin{aligned}
\dot{\bar{P}} &= H_1\bar{P} + \bar{P}H_1' - \bar{P}(I|V)'H'R^{-1}H(I|V)\bar{P} \\
&\quad + \varepsilon^2 \left[\begin{array}{c|c} VQ_2V' & -VQ_2 \\ \hline -Q_2V' & Q_2 \end{array}\right] \\
\bar{P}(0) &= \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & P_z(0) \end{array}\right]
\end{aligned}
\right\} \tag{21}
$$

in which

$$
H_1 = \left[\begin{array}{c|c} A - K_{x_1}H & 0 \\ \hline 0 & C \end{array}\right] \tag{22}
$$

The error covariance matrix, P, for estimators $\hat{\underline{x}}$ and $\hat{\underline{z}}$, defined as

$$
P = \left[\begin{array}{c|c} P_x & P_{xz} \\ \hline P_{xz}' & P_z \end{array}\right] \tag{23}
$$

is computed, using $\bar{P}$, as

$$
P = \bar{P}_x + \left[\begin{array}{c|c} I & V \\ \hline 0 & I \end{array}\right] \bar{P} \left[\begin{array}{c|c} I & V \\ \hline 0 & I \end{array}\right]' \tag{24}
$$

where

$$
\bar{P}_x = \left[\begin{array}{c|c} P_{x_1} & 0 \\ \hline 0 & 0 \end{array}\right] \tag{25}
$$

and

$$
\dot{P}_{x_1} = AP_{x_1} + P_{x_1}A' - P_{x_1}H'R^{-1}HP_{x_1} + Q_1 \quad . \tag{26}
$$

The proof of this theorem, which is patterned after the constructive derivation of Friedland's results given by Mendel and Washburn (Ref. 8), is sketched in Appendix A.

Comments: (1) Equations (15), (19), and (26) comprise a filtering system for the estimation of $\underline{x} \in S$ when colored noise states, $\underline{z}$, do not exist. As such, with $\underline{z}$-states present, $\tilde{\underline{x}}$ is not optimal in any sense, since the true measurement, $\underline{y}(t)$, which contains effects due to $\underline{z}$, is used to obtain $\tilde{\underline{x}}$.

(2) Signal $\hat{\underline{\xi}}$ is an estimate of an n×1 residual noise process, $\underline{\xi}$, which satisfies the following state equation (see Appendix A for derivation):

$$\dot{\underline{\xi}} = (A - K_{x_1}H)\underline{\xi} - \varepsilon V\underline{w} \quad ; \quad \underline{\xi}(0) = \underline{0} \tag{27}$$

Observe that for $\varepsilon = 0$, the unique solution for $\underline{\xi}$ is

$$\underline{\xi}(t) = \underline{0} \quad \forall\, t \geq 0 \tag{28}$$

This result is motivation for a suboptimal series expansion of the Theorem 1 equations about $\varepsilon = 0$, the details of which are given in Section III.

(3) Observe, in Eqs. (17) and (18), that K_z and K_ξ depend on elements P_1 and P_{12} of error covariance matrix $\bar{P}$. It can be shown that

$$P_1 = P_\xi + P_{\xi x_1} + P'_{\xi x_1} \tag{29}$$

and

$$P_{12} = P_{\xi z} + P_{x_1 z} \quad , \tag{30}$$

where x_1 is associated with the artificial system $\dot{\underline{x}}_1 = A\underline{x}_1 + \underline{u}$ and $\underline{y}_a(t) = H\underline{x}_1 + \underline{v}$, in which artificial measurement $\underline{y}_a(t)$ is nonexistent since it is predicated on the artificial assumption that the measurement is not affected by $\underline{z}$. This notion is extremely useful for analysis purposes. We wish to emphasize the fact that $\tilde{\underline{x}}$ and $\hat{\underline{x}}_1$ are quite different; for during the development of $\tilde{\underline{x}}$ we use $\underline{y}(t)$ which is affected by $\underline{z}$, whereas in the development of $\hat{\underline{x}}_1$, we use $\underline{y}_a(t)$ which

is unaffected by $\underline{z}$. The two estimates are related however.

(4) Matrix $\bar{P}$ is of dimension $(n+r) \times (n+r)$; hence, we have been led to the computation of a large-scale matrix Riccati equation, (21), just as we would have been had we followed the usual procedure for obtaining $\hat{\underline{x}}$ and $\hat{\underline{z}}$ by applying the Kalman filter equations to Equations (1) and (2). This obviously represents a serious limitation of our general results, and points out a shortcoming of this multistage decomposition approach for colored noise states. In Section III, we obtain a suboptimal filter that does not require the computation of a large-scale $\bar{P}$ matrix.

(5) For $\varepsilon = 0$, we have the case considered by Tacker and Lee (Ref. 2). It is straightforward to show that, for $\varepsilon = 0$

$$P_1 = 0 \qquad \forall\, t \geq 0 \tag{31}$$

and

$$P_{12} = 0 \qquad \forall\, t \geq 0 \tag{32}$$

This is a direct consequence of Eqs. (28), (29), and (30). In this case, we obtain:

Corollary 1. For $\varepsilon = 0$, the Theorem 1 results reduce to:

$$\hat{\underline{x}} = \tilde{\underline{x}} + V\hat{\underline{z}} \tag{33}$$

and

$$\dot{\hat{\underline{z}}} = C\hat{\underline{z}} + K_z[\underline{y}(t) - HV\hat{\underline{z}} - H\tilde{\underline{x}}] \quad ; \quad \hat{\underline{z}}(0) \tag{34}$$

where the gains are

$$\dot{V} = (A - K_{x_1}H)V + B - VC \quad ; \quad V(0) = 0 \tag{35}$$

and

$$K_z = P_z(HV)'R^{-1} \tag{36}$$

The error covariance matrix P_z is computed from

$$\dot{P}_z = CP_z + P_zC' - P_z(HV)'R^{-1}(HV)P_z \quad ; \quad P_z(0) \tag{37}$$

and

$$P_x = P_{x_1} + VP_zV' \tag{38}$$

$$P_{xz} = VP_z \tag{39}$$

where P_{x_1} still satisfies Eq. (26).

It is of interest to compare our results, in Eqs. (33) - (39), with those in Tacker and Lee (Ref. 2). Their results are limited to a constant C matrix; our C matrix can be time-varying. They also require some extra calculations, which we do not. Their gain matrix V_b [Eq. (12) in Ref. 2] must be computed and inverted to obtain a gain matrix comparable to our matrix V. They also compute a matrix M [Eq. (13) in Ref. 2] which has no physical meaning, and from which they can compute P_x and P_{xz}. A complete comparison between these results is found in Washburn (Ref. 4).

(6) *For $\varepsilon = 0$ and $C = 0$, we obtain the situation considered by* Friedland (Ref. 1) and Mendel and Washburn (Ref. 8). In that case, we obtain:

Corollary 2. For $\varepsilon = 0$ and $C = 0$, the Theorem 1 results reduce to:

$$\hat{\underline{x}} = \bar{\underline{x}} + V\hat{\underline{z}} \tag{40}$$

and

$$\dot{\hat{\underline{z}}} = K_z[\underline{y}(t) - HV\hat{\underline{z}} - H\bar{\underline{x}}] \quad ; \quad \underline{z}(0) \tag{41}$$

where

$$\dot{V} = (A - K_{x_1}H)V + B \tag{42}$$

and

$$\dot{P}_z = -P_z(HV)'R^{-1}(HV)P_z \quad ; \quad P_z(0) \tag{43}$$

The equations for K_z, P_x, and P_{xz} are those given in Corollary 1.

These results are identical to those obtained by Friedland and Mendel and Washburn.

III. A SUB-OPTIMAL SECOND-ORDER MULTISTAGE ESTIMATOR

As discussed in Section II, the exact multistage filtering results for adding colored noise states includes the calculation of $(n+r) \times (n+r)$ matrix $\bar{P}$, where $n+r = \dim(\underline{x},\underline{z})$. In this section we develop a sub-optimal multistage estimator by means of a perturbation technique which can be found in numerous references (Refs. 5, 6, and 7 for example).

To begin, we review the essence of the perturbation technique. Given the differential equation

$$\dot{\Lambda}(t,\varepsilon) = f(\Lambda,\varepsilon,t) \quad ; \quad \Lambda(0) \tag{44}$$

where $t \in [0,T]$ and $0 \leq \varepsilon \leq 1$, and, f is analytic in (Λ,ε,t); then, <u>for sufficiently small ε</u>, the following McClaurin series converges to Λ:

$$\Lambda = \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!} \Lambda_0^{(n)} \quad , \quad t\in[0,T] \tag{45}$$

where

$$\Lambda_0^{(n)} = \frac{d^n}{d\varepsilon^n} \Lambda \bigg|_{\varepsilon = 0} \tag{46}$$

which can be computed using the following system of equations,

$$\dot{\Lambda}_0^{(n)} = \frac{d^n}{d\varepsilon^n} f(\Lambda,\varepsilon,t) \bigg|_{\varepsilon = 0} \quad , \quad n = 0,1,2,\ldots \tag{47}$$

It is difficult to quantify what is meant by "sufficiently small ε," since the bounds needed for ε to be small are functions of the $\Lambda_0^{(n)}$, $n = 0,1,2,\ldots$, and these quantities are very difficult to compute a priori. An example which illustrates the range over which the preceding approximation is valid is given at

the end of this section.

Applying this perturbation technique to our results in Theorem 1, one obtains the following results, which are proven in Appendix B.

Theorem 2. For system S, if $P_{xz}(0) = 0$ and ε is sufficiently small, then sub-optimal multistage minimum-variance filter estimator equations are given by Eqs. (12) - (19), where now

$$P_1 = \varepsilon^2 P_1^2 \tag{48}$$

$$P_{12} = \varepsilon^2 P_{12}^2 \tag{49}$$

and

$$P_z = \bar{P}_z + \varepsilon^2 P_z^2 \tag{50}$$

In Eqs. (48) - (50), P_1^2, P_{12}^2, $\bar{P}_z$, and P_z^2 are computed from the following:

$$\dot{P}_1^2 = (A - K_{x_1} H)P_1^2 + P_1^2(A - K_{x_1} H)' + VQ_2V' \ , \quad P_1^2(0) = 0 \tag{51}$$

$$\dot{P}_{12}^2 = (A - K_{x_1} H)P_{12}^2 + P_{12}^2 C' - P_{12}^2 A_2 \bar{P}_z - P_1^2 A_{12} \bar{P}_z - VQ_2 \ , \quad P_{12}^2(0) = 0 \tag{52}$$

$$\dot{\bar{P}}_z = C\bar{P}_z + \bar{P}_z C' - \bar{P}_z A_2 \bar{P}_z \ , \quad P_z(0) \tag{53}$$

and

$$\dot{P}_z^2 = (C - \bar{P}_z A_2)P_z^2 + P_z^2(C - \bar{P}_z A_2)' - P_{12}^2 A_{12} \bar{P}_z - \bar{P}_z A_{12}' P_{12}^2 + Q_2 \ , \quad P_z^2(0) = 0 \tag{54}$$

where

$$A_1 = H'R^{-1}H \tag{55}$$

$$A_{12} = H'R^{-1}HV \tag{56}$$

and

$$A_2 = (HV)'R^{-1}HV \tag{57}$$

Figure 1 depicts the order of computations for these equations. Observe, that we no longer must solve an $(n+r) \times (n+r)$ Riccati equation. Equations (51), (52), and (54) are Lyapunov-type equations, whereas Eq. (53) is a Riccati equation, but it is only an $r \times r$ equation. At each stage, the quantities shown in the boxes, in Figure 1, can be computed for all $t \in [0,T]$ using quantities at earlier stages.

We refer to our suboptimal estimator as a <u>second-order suboptimal multistage estimator</u> because we have expanded P_1, P_{12}, and P_z in a three-term expansion, like Eq. (45); and, a three-term expansion includes ε^2 terms. Observe that for the colored noise case, as we have defined it, we must go out to at least a second-order approximation. There are no ε-terms in this expansion. This result is a peculiarity of our particular problem. Washburn has considered the application of this perturbation technique to weakly coupled systems, and, in that application first-order expansions are possible (Ref. 4). Observe also that we have a suboptimal <u>multistage</u> estimator rather than a suboptimal estimator because our starting point was Theorem 1 instead of an augmented Kalman-Bucy filter.

<u>Example</u>. Here we present a numerical example which demonstrates the accuracy of our second-order suboptimal multistage estimator. Our system is:

$$S\begin{cases} \dot{x} = -x + z + u \quad ; \quad x(0) & (58) \\ \dot{z} = -z + \varepsilon w & (59) \\ y = x + v & (60) \end{cases}$$

where all signals are scalars, and, $q_1 = q_2 = 1$, and $r = 1/3$.

In order to evaluate the performance of our suboptimal estimator, we computed the exact steady-state error covariances for S and have compared these values with their Theorem 2 counterparts. The exact steady-state error covariances were obtained by solving the nonlinear coupled algebraic Riccati equations for S using an extended Newton-Raphson iteration technique. We obtained the following steady-

state values for P_x, P_{xz} and P_z, from Eqs. (48) - (54):

$$\left.\begin{aligned} P_x &= \frac{1}{3} + \frac{1}{12}\varepsilon^2 \\ P_{xz} &= \frac{1}{6}\varepsilon^2 \\ P_z &= \frac{1}{2}\varepsilon^2 \end{aligned}\right\} \tag{61}$$

Quantities P_x, P_{xz}, and P_z are depicted in Figs. 2, 3, and 4. Both the approximate (solid curves) and exact (dashed curves) results are given as functions of ε. Observe that the second-order results appear to be quite good for ε as large as 1/2.

Our suboptimal multistage filter equations are:

$$\hat{x} = \tilde{x} + \hat{z} + \hat{\xi} \tag{62}$$

$$\dot{\hat{z}} = -\frac{3}{2}\varepsilon^2\hat{z} - \frac{1}{2}\varepsilon^2\hat{\xi} - \frac{1}{2}\varepsilon^2\tilde{x} + \frac{1}{2}\varepsilon^2 y(t) \quad , \quad \hat{z}(0) \tag{63}$$

$$\dot{\hat{\xi}} = -\frac{7}{4}\varepsilon^2\hat{\xi} + \frac{1}{4}\varepsilon^2\hat{z} + \frac{1}{4}\varepsilon^2\tilde{x} - \frac{1}{4}\varepsilon^2 y(t) \quad , \quad \hat{\xi}(0) \tag{64}$$

Of course, $\tilde{x}$ must be computed for the first-order Kalman-Bucy filter equations (15), (19), and (26).

IV. CONCLUSIONS

We have demonstrated that it is indeed possible to extend Friedland's (Ref. 1) bias estimation technique to the problem of estimating dynamical states and colored noise states. We have shown how to obtain a multistage decomposition not only for the state estimation equations, but also for the associated error covariance equations. Additionally, we have related our results to Friedland's and Tacker and Lee's (Ref. 2).

Our exact decomposition, in Theorem 1, can be viewed from a number of points of view. As a structural result, it shows us how estimates of a lower-order system (i.e., $\tilde{x}$) must be modified when colored-noise states are added to the

description of that system. As such, this knowledge can be used to increase our understanding of the interactions between estimates of dynamical and colored noise states. It also suggests possibilities for further approximations, which can be used to reduce the complexity of the exact results. As a computational result, it shows us that there does not seem to be any way to avoid having to solve a larger matrix Riccati equation. As such, the exact decomposition is disappointing; but, we have also shown how to reduce computations (at least with respect to having to solve the larger matrix Riccati equation) by means of a perturbation technique.

One can also view our Theorem 1 results in the context of decentralized estimation theory. Signal $\underline{x}$ is estimated at one level, then $\hat{\underline{z}}$ and $\hat{\underline{\xi}}$ are estimated at the second level, after which coordination takes place at a third level to provide us with $\hat{\underline{x}}$. Sandell, et al. (Ref. 9), mention that in decentralized control, when the coordinator has access to full knowledge (i.e., at each instant of time, each local subsystem transmits instantly and without error its measurements and controls to the coordinator, and the coordinator has perfect recall) his optimal strategy is to cancel the locally computed controls and substitute the global optimal controls. Perhaps this same phenomenon is happening in our Theorem 1 results. This would explain the need for $\hat{\underline{\xi}}$ and the resulting $(n+r) \times (n+r)$ Riccati equation for $\bar{P}$. We are presently studying this conjecture.

Extensions of our results to much more general partitioned dynamical systems can be found in Washburn (4).

Appendix A. Proof of Theorem 1

Our proof of Theorem 1 is patterned after the constructive derivation of Friedland's results, given in Ref. 8, and in Washburn's dissertation (Ref. 4). Due to page limitations, we give only a brief sketch of the proof, since the details are lengthy and can be found in Ref. 4.

Our approach is to assume the existence of the decomposition for $\underline{\hat{x}}$ in Eq. (10), and, to assume that

$$\dot{\hat{\underline{z}}} = B_1\hat{\underline{\xi}} + B_2\hat{\underline{z}} + B_3\bar{\underline{x}} + B_4\underline{y}(t) \quad ; \quad \hat{\underline{z}}(0) \tag{A-1}$$

and

$$\dot{\hat{\underline{\xi}}} = C_1\hat{\underline{\xi}} + C_2\hat{\underline{z}} + C_3\bar{\underline{x}} + C_4\underline{y}(t) \quad ; \quad \underline{\xi}(0) \tag{A-2}$$

where matrices B_1, B_2, B_3, B_4, C_1, C_2, C_3, and C_4 remain to be determined. We then require that $\bar{\underline{x}}$, $\hat{\underline{z}}$, and $\hat{\underline{\xi}}$ be unbiased estimates of $\underline{x}$, $\underline{z}$, and $\underline{\xi}$, respectively. Unbiasedness determines the form of V, in Eq. (16), and it constrains B_1, B_2, and B_3 to be specific functions of B_4, and C_1, C_2, and C_3 to be specific functions of C_4. As such, the requirement for unbiased estimates reduces the number of unknown design matrices in Eqs. (10), (A-1), and (A-2) from nine to two, B_4 and C_4.

In order to perform the unbiasedness analysis just described, it is necessary to obtain the following interesting decomposition of state vector $\underline{x}$:

$$\underline{x} = \Lambda_1\underline{x}_1 + \Lambda_2\underline{z} + \underline{\Lambda}_3 \tag{A-3}$$

where Λ_1, Λ_2, and $\underline{\Lambda}_3$ satisfy certain linear differential equations and $\underline{x}_1$ is associated with the artificial system (see comment 3 after Theorem 1 for further discussions).

$$\dot{\underline{x}}_1 = A\underline{x}_1 + \underline{u} \quad ; \quad \underline{x}_1(0) \tag{A-4}$$

$$\underline{y}_a = H\underline{x}_1 + \underline{v} \tag{A-5}$$

Also needed is the following relationship between non-optimal estimator $\underline{\tilde{x}}$ and optimal estimator of $\underline{x}_1$, $\hat{\underline{x}}_1$:

$$\underline{\tilde{x}} = \hat{\underline{x}}_1 + \bar{\Lambda}_2\underline{z} + \bar{\Lambda}_3\underline{x}_1 + \bar{\underline{\Lambda}}_4 \tag{A-6}$$

where $\bar{\Lambda}_2$, $\bar{\Lambda}_3$, and $\bar{\underline{\Lambda}}_4$ also satisfy certain linear differential equations, which are driven by Λ_1, Λ_2, and $\underline{\Lambda}_3$. It is important to understand the distinction between $\underline{\tilde{x}}$ and $\hat{\underline{x}}_1$. Estimator $\hat{\underline{x}}_1$ is truly optimal; for, it is associated with the artificial system in (A-4) and (A-5), and that system knows nothing about the colored noise states, $\underline{z}$. Measurement $\underline{y}_a$ does not exist, of course, since the real system's-measurement is affected by those states even when colored noise states are suppressed from the real system's state equation. Estimate $\underline{\tilde{x}}$ is obtained for the system $\dot{\underline{x}} = A\underline{x} + B\underline{u}$, $\underline{y} = H\underline{x} + \underline{v}$ — but, $\underline{y}$ is affected by $\underline{z}$.

By means of Eqs. (A-3) and (A-6), it is then possible to express $\underline{x} - \hat{\underline{x}} \triangleq \underline{x}_e$, as

$$\underline{x}_e = (\underline{x}_1 - \hat{\underline{x}}_1) + \nabla(\underline{z} - \hat{\underline{z}}) + (\underline{\xi} - \hat{\underline{\xi}}) \tag{A-7}$$

where, interestingly enough,*

$$\underline{\xi} = \underline{\Lambda}_3 - \bar{\underline{\Lambda}}_4 \tag{A-8}$$

and

$$\nabla = \Lambda_2 - \bar{\Lambda}_2 \tag{A-9}$$

Each of the estimates in (A-7) is an optimal estimate [this would not be the case for $\underline{\tilde{x}}$]. Equation (A-7) is used to study unbiasedness.

To determine gain matrices B_4 and C_4, the trace of the error covariance matrix for $\hat{\underline{x}}$ and $\hat{\underline{z}}$ is minimized, using gradient matrix calculations. This yields Eqs. (17) and (18). Using these optimal values, we also obtain the expressions for $\bar{P}$ and P, given in Eqs. (21) and (24).

We have assumed, a priori, that our estimators are linear. Thus far we have

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* Equation (27) can be obtained from Eq. (A-8) and the differential equations (Ref. 4) which define $\underline{\Lambda}_3$ and $\bar{\underline{\Lambda}}_4$.

not used the fact that our noise processes are gaussian. Using this information, it follows via uniqueness of solution of the matrix Riccati equation that $\hat{\underline{x}}$ and $\hat{\underline{z}}$, obtained via Theorem 1, are the optimal estimators of $\underline{x}$ and $\underline{z}$.

Appendix B. Proof of Theorem 2

Equations (48) - (54) are obtained by applying the Eq. (45) perturbation technique to $\bar{P}$ in Eq. (21). Our counterpart to Eq. (47) is:

$$\left[\begin{array}{c|c} \dot{P}_1^n & \dot{P}_{12}^n \\ \hline \dot{P}_{12}^n & \dot{P}_z^n \end{array}\right] = \frac{d^n}{d\varepsilon^n}\left\{ H_1\bar{P} + \bar{P}H_1' - \bar{P}(I|V)'H'R^{-1}(I|V)\bar{P} + \varepsilon^2 \left[\begin{array}{c|c} VQ_2V' & -VQ_2 \\ \hline -Q_2V' & Q_2 \end{array}\right]\right\}\Bigg|_{\varepsilon = 0} \quad \text{(B-1)}$$

We truncate the infinite-series expansions for P_1, P_{12}, and P_z after three terms, i.e., *

$$P_1 \approx \bar{P}_1 + \varepsilon P_1^1 + \varepsilon^2 P_1^2 \quad \text{(B-2a)}$$

$$P_{12} \approx \bar{P}_{12} + \varepsilon P_{12}^1 + \varepsilon^2 P_{12}^2 \quad \text{(B-2b)}$$

and

$$P_z \approx \bar{P}_z + \varepsilon P_z^1 + \varepsilon^2 P_z^2 \quad \text{(B-2c)}$$

In order to obtain initial conditions for Eq. (B-1), we are led to the following underdetermined set of equations:

$$\left.\begin{array}{l} P_1(0) = \bar{P}_1(0) + \varepsilon P_1^1(0) + \varepsilon^2 P_1^2(0) = 0 \\ P_{12}(0) = \bar{P}_{12}(0) + \varepsilon P_{12}^1(0) + \varepsilon^2 P_{12}^2(0) = 0 \\ P_z(0) = \bar{P}_z(0) + \varepsilon P_z^1(0) + \varepsilon^2 P_z^2(0) = 0 \end{array}\right\} \quad \text{(B-3)}$$

* The factor of ½ which appears in the third term of each expansion has been absorbed into the matrix in that term.

Many different solutions of Eq. (B-3) are possible. We have found that the following solution, which is independent of ε, is very useful:

$$\bar{P}_z(0) = P_z(0) \tag{B-4a}$$

All other initial matrices (e.g., $P_{12}^2(0)$) are zero (B-4b)

A direct consequence of Eq. (B-4b) is, that: $\bar{P}_1(t) = 0$, $P_1^1(t) = 0$, $\bar{P}_{12}(t) = 0$, $P_{12}^1(t) = 0$, and $P_z^1(t) = 0$, $\forall t \geq 0$ [each of these matrices satisfy a homogeneous differential equation, and, for zero initial condition, each matrix is identically zero for $\forall t \geq 0$]; hence, Eqs. (B-2a), (B-2b), and (B-2c) reduce to Eqs. (48), (49), and (50), respectively. Equations (51) - (54) are obtained directly from Eq. (B-1).

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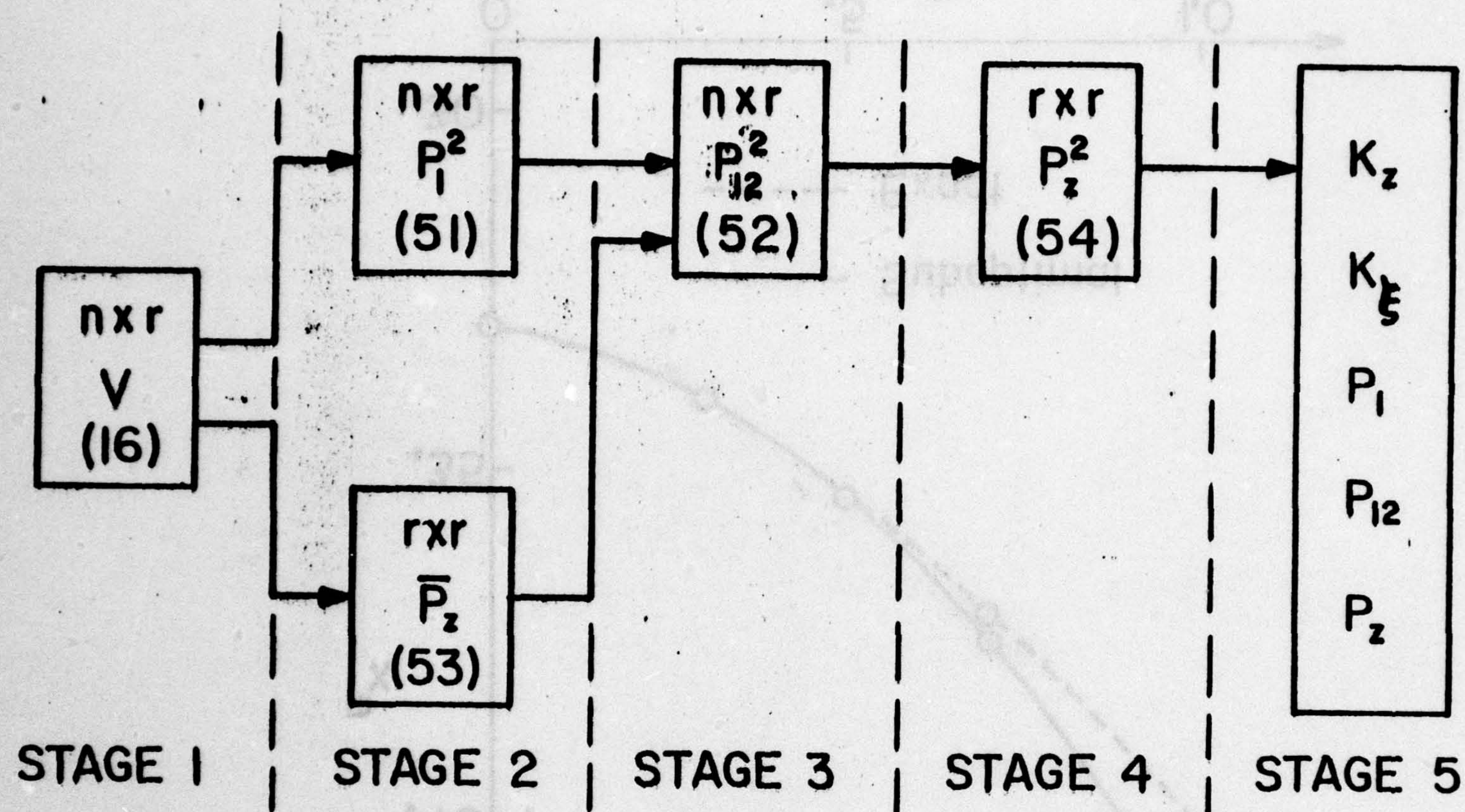
List of Illustrations



nxr
V
(16)
nxr
P_1^2
(51)
rxr
$\bar{P}_z$
(53)
nxr
P_{12}^2
(52)
rxr
P_z^2
(54)
K_z
K_ξ
P_1
P_{12}
P_z
STAGE 1
STAGE 2
STAGE 3
STAGE 4
STAGE 5

FIG. 1

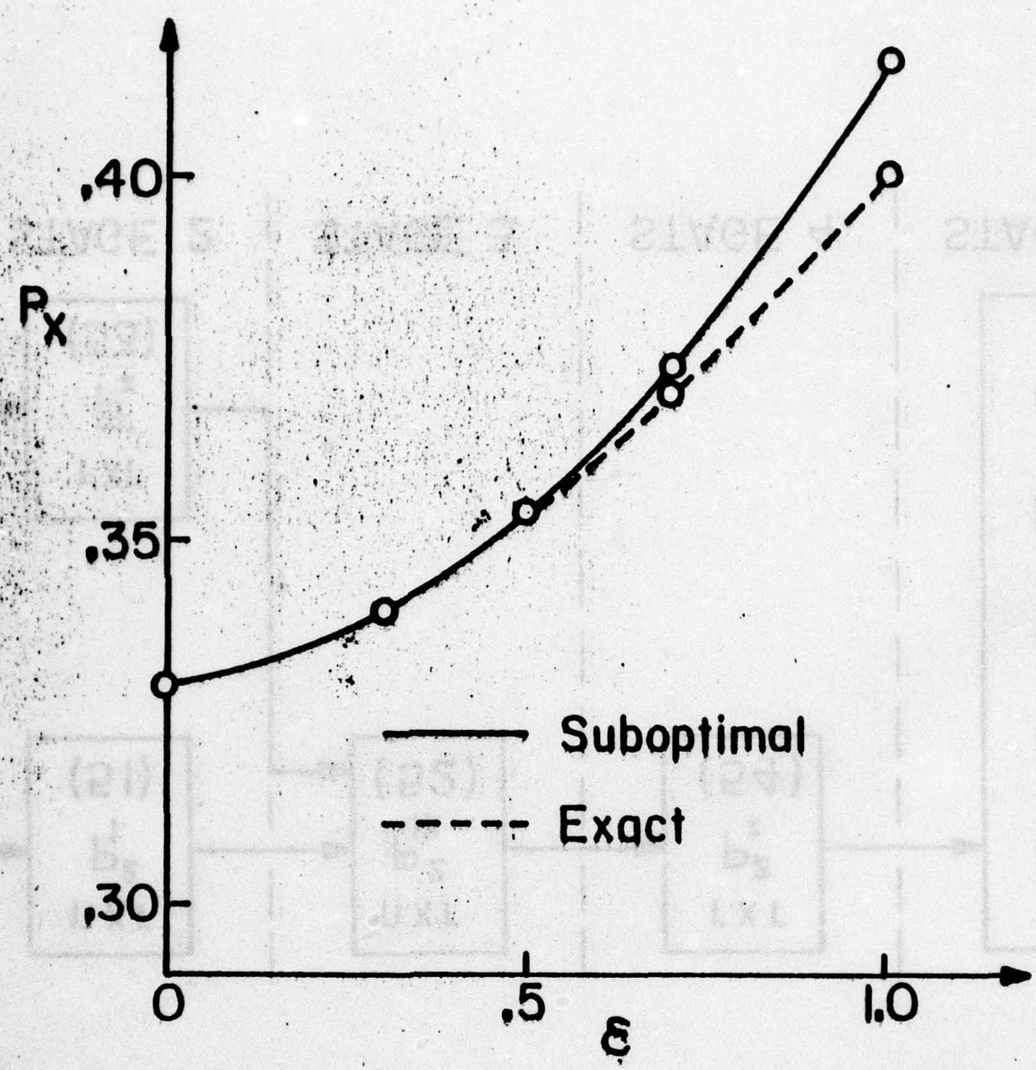
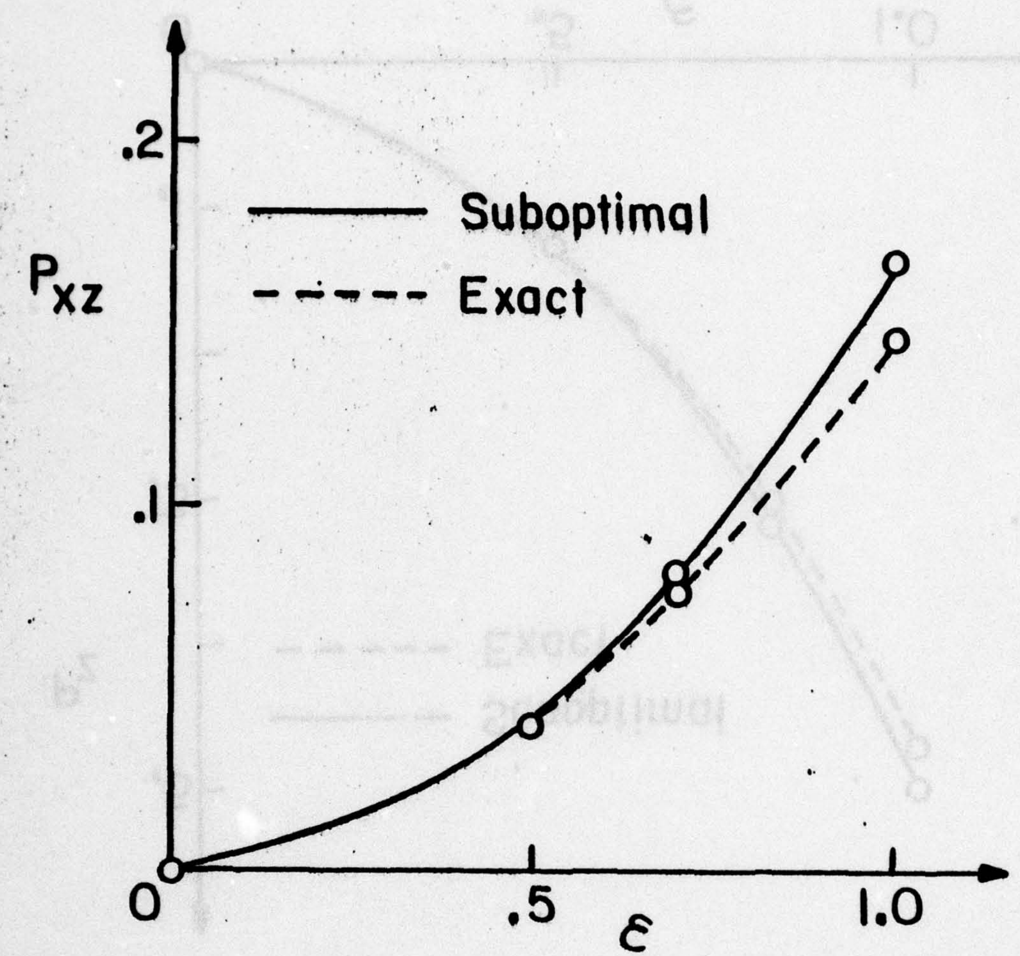


FIG. 2



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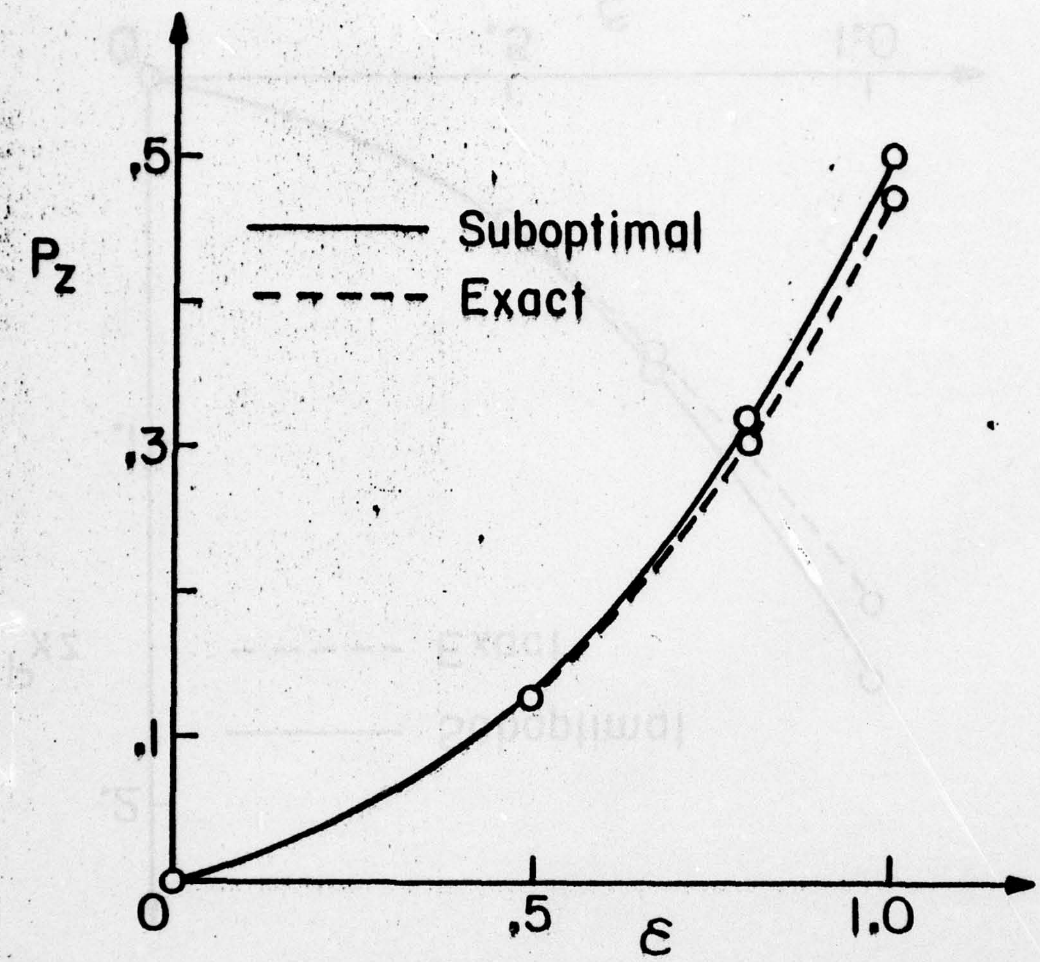

FIG. 4